Lecture 2.

Background

What we track via dynamical systems in models from mathematical ecology: n-tuples, say, for example,

$$(y_1(t), \dots, y_n(t))$$

or

$$(y_1(x,t), \dots, y_n(x,t))$$

$x \in \Omega \subseteq \mathbb{R}^N$. Here $y_i(x,t)$ typically denotes the average population density of some "cohort" or "class" of some biological species at location $x \in \Omega$ and time t, while $y_i(t)$ would represent the average popu tion density over Ω. In some instances, $y_i(t)$

could represent other quantities. For example, $y_i(t)$ could be the total population of the class in Ω. In metapopulation models, it could represent the proportion of Ω occupied or the probability that Ω is occupied by the class in question. In any event, $y_i(t)$ or $y_i(x,t)$ are nonnegative quantities.

There are three basic properties that pertain here:

(i) linear structure

(ii) notion of distance

(iii) partial ordering

Of these, the second requires the most comment at this point. In the case of tuples

$(y_1(t), \ldots, y_n(t))$ and $(y_1^*(t), \ldots, y_n^*(t))$, the most familiar notion of distance is the Euclidean; namely,

(2.1) $$d((y_1(t), \ldots, y_n(t)), (y_1^*(t), \ldots, y_n^*(t))) = \sqrt{\sum_{i=1}^{n} (y_i(t) - y_i^*(t))^2}.$$

It is extremely well-known that:

(i) $d((y_1(t), \ldots, y_n(t)), (y_1^*(t), \ldots, y_n^*(t))) \geq 0$

with $d((y_1(t), \ldots, y_n(t)), (y_1^*(t), \ldots, y_n^*(t))) = 0$

$\Leftrightarrow$ $y_i(t) = y_i^*(t)$ for $i = 1, \ldots, n$.

(ii) $d((y_1(t), \ldots, y_n(t)), (y_1^*(t), \ldots, y_n^*(t)))$

$= d((y_1^*(t), \ldots, y_n^*(t)), (y_1(t), \ldots, y_n(t)))$

(iii) If $(y_1^{**}(t), \ldots, y_n^{**}(t))$ is any other n-tuple,

$d((y_1(t), \ldots, y_n(t)), (y_1^*(t), \ldots, y_n^*(t)))$

$\leq d((y_1(t), \ldots, y_n(t)), (y_1^{**}(t), \ldots, y_n^{**}(t)))$

$+ d((y_1^{**}(t), \ldots, y_n^{**}(t)), (y_1^*(t), \ldots, y_n^*(t)))$ (triangle inequality)

For the sake of simplicity of expression, let's let $y(t) = (y_1(t), \ldots, y_n(t))$ and $y^*(t) = (y_1^*(t), \ldots, y_n^*(t))$.

Since we have endowed the set of n-tuples with a linear structure, we can think of $y(t) - y^*(t)$ as a vector in its own right or as the line segment joining the vectors $y(t)$ and $y^*(t)$.

We have $d(y(t), y^*(t)) = d(y(t) - y^*(t), 0)$.

This leads us to define

$$\|y(t)\| = d(y(t), 0)$$

as the <u>Euclidean norm</u> of $y(t)$ (i.e. distance to $(0, \ldots, 0)$). It is immediate that

(iv) $\|y\| \geq 0$ with $\|y\| = 0$ only if $y_i = 0$ for $i = 1, \ldots, n$

(v) $\|cy\| = |c| \, \|y\|$ for any $c \in \mathbb{R}$

(vi) $\|y + y^*\| \leq \|y\| + \|y^*\|$

Any function from $\mathbb{R}^N$ into $[0,\infty)$ that satisfies (iv) - (vi) is of course called a norm on $\mathbb{R}^N$. Let $\| \ \|_1$ denote such a function. Then $d_1(x,y) = \|x-y\|_1$ defines a metric or distance function satisfying (i) - (iii). As such, $\mathbb{R}^N$ may be viewed as a <u>metric space</u> and topologized with basic open sets given by

$$B(x,\varepsilon) = \{ y \in \mathbb{R}^N \mid \|x-y\| < \varepsilon \}$$

where $\varepsilon > 0$.

It is a fundamental fact that for any such $\|\cdot\|_1$ there are positive constants α, β so that

(2.2) $$\alpha \|y\|_1 \leq \|y\| \leq \beta \|y\|_1$$

where $\|\cdot\|$ is our original Euclidean norm on $\mathbb{R}^N$.

It follows from (2.2) that the metric space topologies induced by any two norms on $\mathbb{R}^N$ are equivalent. Moreover, given any sequence of points $\{y_n\} \subseteq \mathbb{R}^N$ with the property that given $\varepsilon > 0$, there is a $K \in \mathbb{Z}^+$ such that $\|y_n - y_m\| < \varepsilon$ for $n, m \geq K$, it follows from Bolzano-Weierstrass that there is a $y_0 \in \mathbb{R}^N$ so that if $\varepsilon > 0$ is given, there is an $M \in \mathbb{Z}^+$ so that

$$\|y_n - y_0\| < \varepsilon$$

if $n \geq M$. Sequences with the first property are called Cauchy while sequences with the second are called convergent. We call a metric space in which Cauchy sequences converge a complete metric space. A linear space that is equipped by a norm is called a normed linear space. If said normed linear space is complete as a metric space, it is called a Banach space. Banach spaces will be the basic mathematical objects of interest to our discussion.

When we now consider space explicitly via $(y_1(x,t),\ldots,y_n(x,t))$, we face the issue of what we mean by saying $y(x,t)$ and $y^*(x,t)$ are close to each other. Notice that we now consider n-tuples of real-valued functions of space and time, so that the underlying linear structure is infinite dimensional. Should having $y(x)$ close to $y^*(x)$

mean that $y(x)$ is close to $y^*(x)$ in some sense at every point $x \in \Omega$ or could it mean that $y(x)$ is close to $y^*(x)$ in some averaged sense, such as having

$$\int_\Omega |y(x) - y^*(x)|\, dx$$

small? Should only the values of $y(x)$ and $y^*(x)$ matter, or should we take into account their spatial derivatives up to some order, and if so, to what order? There are myriad possibilities. The notion of complete metric space gives us enough generality to accomodate any of the possibilities. However, since we now consider infinite dimensional linear spaces, the metric spaces that arise from norms are no longer automatically topologically equivalent. Indeed, what configurations $y(x,t)$ we must consider to have a complete metric space depends on the norm and consequent metric.

Let's illustrate via an example. Let $\Omega = [0,2]$ and suppose we are considering a model for a single species on $[0,2]$. One family of continuous population densities could be

$$f_k(x) = \begin{cases} x^k & 0 \le x \le 1 \\ 1 & 1 \le x \le 2 . \end{cases}$$

Clearly these densities are all continuous. However, the pointwise limit is easily seen to be

$$f_0(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & 1 \le x \le 2 \end{cases}$$

which is not. If we take $\|f\|_\infty = \max\limits_{x \in [0,2]} |f(x)|$, we see that

$$\|f_n - f_{n+k}\| = \max_{x \in [0,2]} |f_n(x) - f_{n+k}(x)| = \max_{x \in [0,1]} |x^n - x^{n+k}|$$

$$= \left(\frac{n}{n+k}\right)^{\frac{n}{k}} \left[1 - \frac{n}{n+k}\right]$$

$$\to 1 \quad \text{as } k \to \infty \quad \text{for any fixed } n.$$

So our family simply fails to be Cauchy (It is well-known that the limit of a Cauchy sequence of continuous functions under the $\|\cdot\|_\infty$ norm is continuous.)

On the other hand, if we take $\|f\|_1$

$$= \int_0^2 |f(x)|\,dx, \quad \|f_n - f_{n+k}\| = \int_0^1 (x^n - x^{n+k})\,dx$$

$$= \frac{1}{n+1} - \frac{1}{n+k+1} = \frac{n+k+1-(n+1)}{(n+1)(n+k+1)} = \frac{k}{(n+1)(n+k+1)}$$

$< \frac{1}{n+1}$ for all k. So the sequence is Cauchy under the $\|\cdot\|_1$. However, we must allow discontinuous functions to have a complete metric space in this case.

As noted, Banach spaces are the primary objects of study in our discussion. Since we are thinking of n-tuples of population density profiles over the focal habitat patch in question, we will often restrict ourselves to points in the space which are componentwise nonnegative. We will usually want the set of points we consider to be closed. A fundamental fact from functional analysis is that <u>closed subsets of Banach spaces are complete metric spaces</u>.

As noted in Lecture 1, in these lectures, we will assume $\frac{\partial u_i}{\partial t}$ does not depend explicitly upon time; i.e., that the growth and dispersal law is <u>autonomous</u>. In such case, the time evolution of species densities so governed is uniquely determined for all time $t \geq 0$ by the initial configuration of densities; i.e.

(2.3) $$(u_1(x,t), \ldots, u_n(x,t)) = \varphi(u_1(x,0), \ldots, u_n(x,0), t)$$

for $t \geq 0$.

Here φ is continuous when viewed as a function from

$$Y \times [0, \infty) \text{ into } Y$$

where the complete metric space Y is some suitable prespecified collection of n-tuples of possible species densities on the underlying habitat $\bar{\Omega}$.

Moreover, we have

(2.4) $$\varphi(u_1(x,0), \ldots, u_n(x,0), t+t')$$
$$= \varphi[\varphi(u_1(x,0), \ldots, u_n(x,0), t), t']$$

(the configuration to which $(u_1(x,0), \ldots, u_n(x,0))$ evolves after $t+t'$ units of time ($t, t' > 0$) is the same configuration as arises when

$\varphi(u_1(x,0), \dots, u_n(x,0), t)$ is viewed as an initial configuration and then evolves for t' units of time.)

Autonomous reaction-diffusion models are examples of <u>continuous-time semi-dynamical systems</u>,

<u>Basics of Dynamical Systems</u>.

(i) A continuous-time dynamical system or flow is a continuous function $\pi : Y \times \mathbb{R} \to Y$ where Y is a metric space so that

$$\pi(u, 0) = u$$

$$\pi(u, t+t') = \pi(\pi(u,t), t')$$

for all $t, t' \in \mathbb{R}$ so that $\pi(u, t+t')$ and $\pi(u,t)$ are defined. (When the set of real numbers $\mathbb{R}$ in the preceding is replaced by $[0,\infty)$,

π is called a semi-dynamical system or semi-flow.

(ii) If π is a dynamical system and $u \in Y$, $\pi(u,t)$ exists for all t in some maximal interval $(t_-(u), t_+(u))$ with $-\infty \le t_-(u) < 0 < t_+(u) \le \infty$. The set of points

$$\gamma(u) = \{\pi(u,t) : t_-(u) < t < t_+(u)\}$$

is called the <u>orbit of π through</u> u.

The set of points

$$\gamma^+(u) = \{\pi(u,t) : 0 \le t < t_+(u)\}$$

is called the <u>positive semi-orbit of π through</u> u.

(If π is a semi-dynamical system that does not extend to be a dynamical system, only positive semi-orbits are guaranteed to exist for all u.)

(iii) In order to consider long-term or asymptotic behavior of a dynamical or semi-dynamical system, it is legitimate to require at a minimum that

(2.5) $$t_+(u) = \infty$$

for all $u \in Y$. Not all dynamical or semi-dynamical systems have this property.

If $$\frac{dy}{dt} = (y-K)^2, \quad y(0) = m > K$$

with $K > 0$, then $y = K + \dfrac{(m-k)}{1-(m-k)t}$

$\to +\infty$ as $t \to \left(\dfrac{1}{m-k}\right)^-$.

Most ecological models (at least nonlinear ones) do not envision orbits that "blow up" in finite time. They include mechanisms which preclude unbounded growth in species densities in the long term.

So we actually impose conditions that restrict systems much more stringently than just (2.5). We require that systems be <u>dissipative</u>; there is a bounded subset U of Y so that for any $u \in Y$, $\pi(u,t) \in U$ for sufficiently large t. (In a Banach space, U bounded means there is an M so that if $v \in U$, $\|v\| \leq M$). For an individual element, how large t must be for $\pi(u,t) \in U$ is allowed to depend on u.)

Note that in the context of dissipative systems of ordinary differential equations, if $(y_1^0, \ldots, y_n^0)$ is any initial configuration and t_j is any sequence of times with $t_j \to \infty$ as $j \to \infty$

Bolzano-Weierstrass implies $\pi((y_1^0, \cdots, y_n^0), t_{j_\ell})$ converges to some point $(\bar{y}_1, \cdots, \bar{y}_n)$ for some subsequence $\{t_{j_\ell}\}$ of $\{t_j\}$. In other words, $\pi((y_1^0, \cdots, y_n^0), t_j)$ is pre-compact. (A set P is pre-compact if any sequence of points in the set has a convergent subsequence. If the limit always lies in P, P is compact.)

So in o.d.e. systems, dissipativity guarantees the pre-compactness of positive semi-orbits.

In the reaction-diffusion case, dissipativity by itself does not guarantee such. Rather, dissipativity must be used in conjunction with the smoothing action associated to the elliptic operators in the system to draw this conclusion. (Lectures 3 & 4)

(iv) If we want to interpret

$$y_i' = f_i(y_1, \ldots, y_n)$$

as an ecological model, we need for $\pi((y_1^0, \ldots, y_n^0), t)$ to be componentwise nonnegative for all $t \geq 0$.

The "Kolmogorov condition

$$f_i(y_1, \ldots, y_n) = y_i \hat{f}_i(y_1, \ldots, y_n)$$

guarantees that such is the case by the uniqueness of solutions to initial value problems.

(v) If (Y, π) is a dissipative dynamical or semi-dynamical system, a subset U of Y is forward invariant under π if $\gamma^+(u) \subseteq U$ for all $u \in U$. If $t_-(u) = -\infty$ and $\gamma(u) \subseteq U$ for all $u \in U$, U is invariant under π.

(vi) If for $u \in Y$, $\gamma^+(u)$ is pre-compact, then there is a bounded set in Y which attracts points on $\gamma^+(u)$.

This set is the ω-limit set of u, $\omega(u)$, with

$$\omega(u) = \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} \{\pi(u,r) : r \geq s\}}$$

($\omega(u)$ consists of all limits of all sequences $\{\pi(u,t_n) \mid t_n \to \infty\}$)

$\omega(u)$ is
(a) nonempty
(b) compact
(c) connected
(d) invariant under π

(vii) The alpha limit set of u, $\alpha(u)$, consists of all limits of all sequences $\{\pi(u,t_n) \mid t_n \to -\infty\}$.

(viii) If now, for a set U, the set $\{\pi(u,r) : u \in U, r \geq s\}$ is pre-compact for some $s \geq 0$, there is a companion notion of the omega limit set of U, $\omega(U)$, with

$$\omega(U) = \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} \{\pi(u,r) : u \in U, r \geq s\}}$$

($\omega(U)$ is the set of all limits of all sequences $\{\pi(u_n,t_n) \mid u_n \in U, t_n \to \infty\}$)

$\bigcup_{u \in U} \omega(u) \subseteq \omega(U)$, but in general $\omega(U)$ is much larger. However, we use $\bigcup_{u \in U} \omega(u)$ in several places in our discussion. For the sake of simplicity of notation, we will use $\omega(U)$ to denote this set, but make clear that this use is non-standard.

(ix) A global attractor for a dynamical or semi-dynamical system π on (Y, d) is a set $\mathcal{A}$ which is

(a) compact

(b) invariant under π

(c) such that for all bounded subsets V of Y,

$$\lim_{t \to \infty} \sup_{v \in V} \inf_{u \in \mathcal{A}} d(\pi(v,t), u) = 0$$

(x) A fundamental result of Bilotti and LaSalle (1971) says that if (Y, d) is complete, π is dissipative, and for all $t > t_0 \geq 0$, the set $\pi(U, t)$ is pre-compact if U is bounded (i.e., $\pi(\cdot, t) : Y \to Y$ is compact), then π has a nonempty global attractor.

If $\mathcal{A}$ is a global attractor and $\varepsilon > 0$ is given, then $\mathcal{B}(\mathcal{A}, \varepsilon) = \{ y \in Y : d(y,u) < \varepsilon$ for some $u \in \mathcal{A} \}$ is such that if $V \subseteq Y$ is bounded, then

$$\pi(V, t) \subseteq \mathcal{B}(\mathcal{A}, \varepsilon)$$

for all $t \geq t_0$, where $t_0 = t_0(V)$.

(xi) A point $u \in Y$ such that $\pi(u,t) = u$ for all $t \in \mathbb{R}$ is called an equilibrium or equilibrium point.

(Most basic example of a global attractor is a globally attracting equilibrium)

(xii) The notion of a monotone dynamical or semi-dynamical system is a possibility when Y admits a partial ordering.

In the examples we consider Y is a closed subset of a Banach space and the partial ordering is induced by a closed subset K called a positive cone via

$$u_1 \leq u_2 \iff u_2 - u_1 \in K$$

K satisfies:

(2.6) (i) $cu \in K$ if $u \in K$ and $c \geq 0$

(ii) $u_1 + u_2 \in K$ if $u_1, u_2 \in K$

(iii) $u, -u \in K \Rightarrow u = 0$

(xiii) Once a partial ordering has been established in Y, π will be monotone provided

$$u_1 \leq u_2 \text{ in } Y \Rightarrow \pi(u_1, t) \leq \pi(u_2, t) \text{ in } Y$$

for all $t \neq 0$ (when π is a dynamical system) or $t > 0$ (when π

is a semi-dynamical system)

(xiv) A positive semi-orbit $\gamma^+(u)$ of π is said to be increasing provided $\pi(u,t_1) \le \pi(u,t_2)$ for $0 \le t_1 < t_2$, with an analogous definition of decreasing ~~pos~~ positive semi-orbit.

Suppose π admits an increasing positive semi-orbit $\gamma^+(u_1)$ and a decreasing positive semi-orbit $\gamma^+(u_2)$ with $u_1 \le u_2$. Then if $u_1 \le u \le u_2$, the positive semi-orbit $\gamma^+(u)$ lies above $\gamma^+(u_1)$ and below $\gamma^+(u_2)$. Moreover, if positive semi-orbits are precompact, an important result in monotone dynamical systems theory (e.g. Smith (1995))

$\Rightarrow$ $\omega(u_1)$ and $\omega(u_2)$ are equilibria for π.

$\pi(u_1,t) \le \pi(u_2,t) \Rightarrow \omega(u_1) \le \omega(u_2)$. Thus

$\omega(u) \subseteq \{v \mid \omega(u_1) \leq v \leq \omega(u_2)\} = [\omega(u_1), \omega(u_2)]$

(order interval) for all $u \in [u_1, u_2]$.

In the context of reaction-diffusion models for interacting biological species in isolated bounded habitat patches, positive semi-orbits that originate at configurations which are so-called lower solutions to the corresponding steady-state elliptic system are increasing; those originating from upper solutions to the elliptic system are decreasing. (Aronson and Weinberger)

(xv) Utility of monotonicity and comparison extends beyond systems with partial orderings via so-called practical persistence approaches.

(xvi) A discrete time dynamical system Π on a complete metric space Y is a continuous function

$\pi : Y \times \mathbb{Z} \to Y$ so that

$$\text{(i)} \quad \pi((u,0)) = u \qquad \forall\, u \in Y$$

$$\text{(ii)} \quad \pi(u, k+k') = \pi(\pi(u,k), k')$$

for all $k, k' \in \mathbb{Z}$ so that $\pi(u, k+k')$ and

$\pi(u,k)$ are defined.

(If π is to be a semi-dynamical system,

$\mathbb{Z}$ is replaced with $\mathbb{Z}^+ \cup \{0\}$)